

Ladder Climbing and Autoresonant Acceleration of Plasma Waves

I. Barth^a, I. Y. Dodin^{a,b}, and N. J. Fisch^{a,b}

^aPrinceton Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08543, USA

^bDepartment of Astrophysical Sciences, Princeton University, Princeton, New Jersey 08540, USA

(Dated: July 22, 2015)

When the background density in a bounded plasma is modulated in time, discrete modes become coupled. Interestingly, for appropriately chosen modulations, the average plasmon energy might be made to grow in a ladder-like manner, achieving up-conversion or down-conversion of the plasmon energy. This reversible process is identified as a classical analog of the effect known as quantum ladder climbing, so that the efficiency and the rate of this process can be written immediately by analogy to a quantum particle in a box. In the limit of densely spaced spectrum, ladder climbing transforms into continuous autoresonance; plasmons may then be manipulated by chirped background modulations much like electrons are autoresonantly manipulated by chirped fields. By formulating the wave dynamics within a universal Lagrangian framework, similar ladder climbing and autoresonance effects are predicted to be achievable with general linear waves in both plasma and other media.

PACS numbers: 52.35.-g, 52.35.Mw, 42.65.-k, 47.10.Df

Introduction. — Quantum mechanics is well known to be closely related to the mechanics of classical waves [1–3]. This permits applying common techniques for manipulating quantum and classical systems and helps bridging seemingly different areas of physics. One important technique to study in this context is ladder climbing (LC), which is the successive transfer of quanta through nonequally-spaced energy levels due to an oscillating driving force with chirped frequency [4, 5]. The system energy changes with time in a ladder-like manner during LC, with each transition described by the famous Landau-Zener (LZ) theory [6]. In the limit of continuous spectra, the effect has been widely known as classical autoresonance (AR), enjoying numerous applications in physics of plasmas [7, 8], fluids [9], Josephson junctions [10], optics [11], and even planetary dynamics [12]. In contrast, the discrete nature of LC is visible only in systems with sufficiently discrete spectra and, so far, has been studied exclusively in quantum contexts [4, 5, 13–15]. Whether classical systems can exhibit LC has remained an open question.

Here we report the first theoretical prediction of LC in a classical system, namely, in an ensemble of plasma waves. For simplicity, we consider one-dimensional collisionless plasma, with nondissipative Langmuir waves, whose spectrum is quantized due to the boundary conditions. We derive a Schrödinger-type equation for the “plasmon wave function”, which is a classical measure of the electric field whose norm (the total wave action) is manifestly conserved under mode coupling. This system is mathematically equivalent to a quantum particle in electrostatic potential. Hence, plasmons can be manipulated by resonant modulation of the underlying medium, much like electrons and molecules are manipulated by resonant external fields [16, 17]. In particular, we show that plasmons can exhibit both LC and AR and can be controllably transported up and down in momen-

tum space. Finally, we report a unifying Lagrangian formulation of the problem that paves the way for applying these techniques to general classical waves.

Basic equations. — For simplicity, we consider an electron plasma described by a hydrodynamic model. The equations for the electron density n_e , electron flow velocity u_e , and the electric field E are then as follows:

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0, \quad (1)$$

$$\frac{\partial}{\partial t}(n_e u_e) + \frac{\partial}{\partial x}(n_e u_e^2) = -\frac{e}{m_e} n_e E - \frac{1}{m_e} \frac{\partial P}{\partial x}, \quad (2)$$

$$\frac{\partial E}{\partial x} = -4\pi e(n_e - Z n_i). \quad (3)$$

Here, $-e$ and m_e are the electron charge and mass, P is the electron pressure, Ze is the ion charge, and n_i is the ion density. We neglect high-frequency oscillations of n_i and consider n_i to be a slow function, $Z n_i = n_0 + n_d(t, x)$; here n_0 is the unperturbed electron density, and n_d is a prescribed driving modulation. Such a modulation can be created by external fields, e.g., by means of ponderomotive forces. Then, $n_e = n_0 + n_d(t, x) + n(t, x)$, where $n \ll n_0$ determines a small uncompensated charge density due to electron inertia. For simplicity, we adopt an isentropic model, $P = P(n_e) \approx P_0 + 3m_e v_{th}^2 (n + n_d) + R(n + n_d)^2/2$; here v_{th} is the electron thermal speed, and $R = \partial^2 P / \partial n^2|_{n_0}$, which are considered constant. We assume hard-wall boundary conditions, so $u_e|_{x=0} = u_e|_{x=\ell} = 0$, where ℓ is the plasma length. We also assume $E|_{x=0} = E|_{x=\ell} = 0$, so any field is representable as a series of $\sin(k_m x)$, where $k_m = \pi m / \ell$. Then, according to Eqs. (1)–(3), the boundary conditions for the density must be $\partial_x n|_{x=0} = \partial_x n|_{x=\ell} = 0$, so n is a series of $\cos(k_m x)$. We consider the external driving modulation to be the N th standing-wave mode,

$$n_d(x, t) = n_0 A \cos(k_N x) \cos \varphi_d(t). \quad (4)$$

We assume that $A \ll 1$ and $\omega_d = \dot{\varphi} \ll \omega_p$, where $\omega_p^2 = 4\pi e^2 n_0 / m_e$. Hence, Eqs. (1)-(3) can be combined in a single equation. Specifically, let us subtract the spatial derivative of Eq. (2) from the temporal derivative of Eq. (1); then substitute $\partial_x E = -4\pi en$, which flows from Eq. (3), and integrate over x . Assuming that n is small enough, we neglect terms nonlinear in $E = O(n)$, and we also neglect the slow driving force, $O(E^0 A^1)$, as nonresonant to the rapid plasma oscillations. As we estimate below, resonant terms of higher orders in A can be dropped too. To the lowest order in ω_d [18], this gives the following linear dimensionless equation for E ,

$$\frac{\partial^2 E}{\partial t^2} - 3 \frac{\partial^2 E}{\partial x^2} + E = -\tilde{n}_d E + \tilde{R} \frac{\partial}{\partial x} \left(\tilde{n}_d \frac{\partial E}{\partial x} \right). \quad (5)$$

Here and further we measure time in units ω_p^{-1} and length in units v_{th}/ω_p ; also, $\tilde{n}_d = n_d/n_0$, and $\tilde{R} = Rn_0/(mv_{th}^2)$. Next, we decompose the field into unperturbed eigenmodes,

$$E = \text{Re} \sum_{m=1}^{\infty} E_m e^{-i\omega_m t} \sqrt{2/\ell} \sin(k_m x), \quad (6)$$

where $E_m(t)$ are complex coefficients. To zeroth order in A , Eq. (5) yields the dimensionless dispersion relations $\omega_m^2 = 1 + \beta m^2$, where $\beta = 3\pi^2/\ell^2$ is analogous to the anharmonicity parameter in a quantum oscillator [5]. We also assume $\beta m^2 \ll 1$, as usual. It is convenient to introduce new variables ψ_m via $\rho_m \psi_m(t) = E_m e^{-i\omega_m t}$. Here ρ_m are constants such that in the unperturbed system, $|\psi_m|^2$ are the actions of individual modes; specifically one finds $\rho_m = (8\pi/\omega_m)^{1/2}$ [19]. The equations for ψ_m , obtained via Fourier-transforming Eq. (5), are:

$$i\dot{\psi}_m = \omega_m \psi_m + \sum_{m'} h_{m,m'} \psi_{m'}, \quad (7)$$

$$h_{m,m'} = \frac{\rho_m \rho_{m'}}{8\pi\ell} \int_0^\ell \sin(k_m x) \hat{\mathcal{F}} \sin(k_{m'} x) dx, \quad (8)$$

where $\hat{\mathcal{F}} = \tilde{n}_d - \partial_x \tilde{R} \tilde{n}_d \partial_x$ is a differential operator. Since $h_{m,m'}$ is Hermitian, the evolution of ψ_m is manifestly unitary; i.e., the wave total action $\sum_m |\psi_m|^2$ is conserved. The vector $\psi = (\psi_1, \psi_2, \dots)$ can then be understood as the plasmon wave function in the energy representation. Next, we will assume the resonance condition $\omega_d \approx \omega_{m,m+N}$, where the level spacing is $\omega_{m,m+N} = \omega_{m+N} - \omega_m \approx \beta N(m + N/2)$. Then, $\beta \sim \omega_d \ll 1$, and therefore, in the already small coupling term, we must adopt $\omega_m \approx 1$, i.e., $\rho_m \approx \sqrt{8\pi}$ and neglect the higher order terms in pressure. This leads to

$$i\dot{\psi}_m = \omega_m \psi_m + \frac{A}{4} (\psi_{m-N} + \psi_{m+N}) \cos \varphi_d. \quad (9)$$

Note also that, for $\beta m^2 \ll 1$, Eq. (9) is equivalent to the energy representation of the Schrödinger equation for a

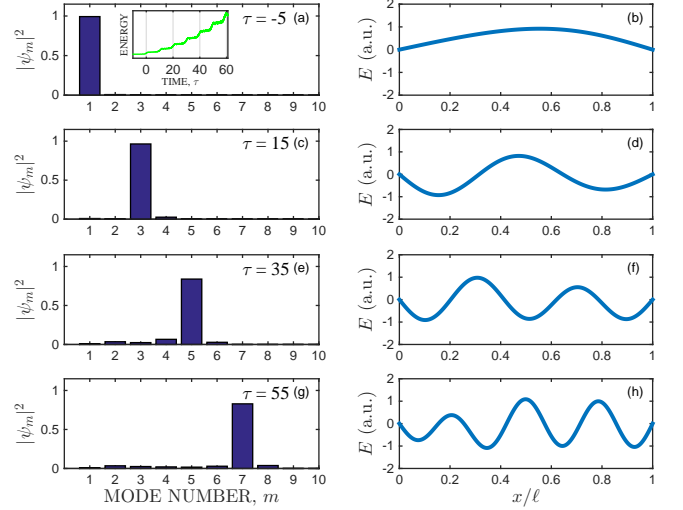


FIG. 1: (color online) Langmuir wave evolution in the LC regime ($P_2 = 10$). Left – spectral representation; snapshots of the occupation numbers, $|\psi_m|^2$, vs mode numbers, m at times $\tau = [-5, 15, 35, 55]$. Right – spatial representation; snapshots of the wave field, E , vs x at the same times. The inset in subplot (a) shows the energy growing with time in a ladder-like manner, where the transitions occur at the theoretically predicted times $\tau_m = mP_2$.

quantum particle in a square potential well [20]. This can be understood as follows: at weak spatial dispersion, all ω_m are close to ω_p , so Eq. (5) permits a quasioptical approximation, turning into the standard quantum Schrödinger equation with \tilde{n}_d serving as an effective potential. (This analogy has been noted, e.g., in Ref. [21].)

LC regime. — Classical AR was previously studied in the infinite square potential well as a limiting case ($j \rightarrow \infty$) of the potential $V_0 = x^{2j}$ [22], but quantum LC was not [23]. For studying LC in our system, we consider the modulation (4) with $N = 1$ and a monotonically increasing driving frequency, $\omega_d = \omega_{1,2} + \alpha t$, where $\alpha > 0$. At $t = 0$, this will initiate resonant transitions between levels $m = 1$ and $m = 2$, which we denote as $1 \rightarrow 2$. Later, transitions between higher levels, $m \rightarrow m + 1$, occur when the resonance condition, $\omega_d = \omega_{m,m+1}$, is satisfied. Following the quantum LC theory [4, 5], we define slow time $\tau = \sqrt{\alpha} t$, driving parameter $P_1 = A/(4\sqrt{\alpha})$, and anharmonicity parameter, $P_2 = \beta/\sqrt{\alpha}$, which plays the role of an effective Planck constant in the classical system. If $P_2 \gg 1 + P_1$, the system remains in the quantum LC regime, when only two levels are resonantly coupled at any given time. Then, transitions $m \rightarrow m + 1$ occur at times $\tau_m = mP_2$, where each such transition can be described by the commonly known LZ theory [2, 6] and thus has a probability

$$\mathcal{P}_{m \rightarrow m+1} = 1 - \exp(-\pi P_1^2/2). \quad (10)$$

At large enough P_1 , $\mathcal{P}_{m \rightarrow m+1} \approx 1$, i.e., all quanta are being transferred, and the dynamics in this limit is char-

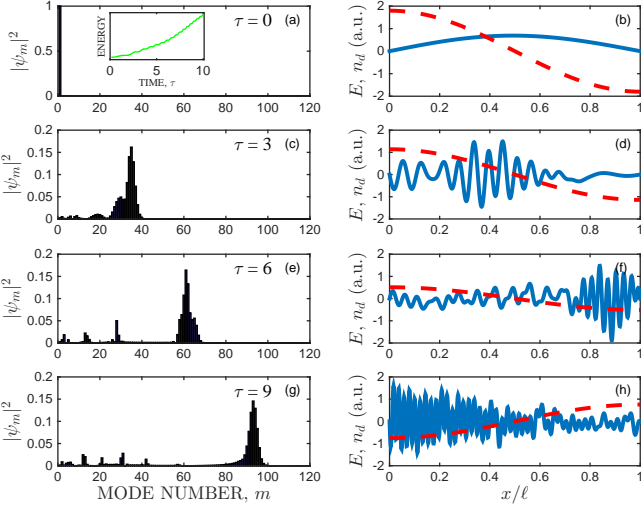


FIG. 2: (color online) Langmuir wave evolution in the AR regime ($P_2 = 0.1$). Left – spectral representation; snapshots of the occupation numbers, $|\psi_m|^2$, are shown vs mode numbers, m , at times $\tau = [0, 3, 6, 9]$, illustrating an autoresonant plasmon trapped in a density modulation effective potential. Right – spatial representation; snapshots of the wave field, E (solid blue lines), and the driving density modulations, n_d (dashed red lines), are shown vs x at the same times. The inset in subplot (a) shows the energy growing with time.

acterized by successive two-level LZ transitions.

In Fig. 1, we illustrate this LC dynamics of Langmuir modes by numerically simulating Eq. (9) with $N = 1$ and initial conditions $\psi_m(t = t_0) = \delta_{m,1}$. We choose $\{\alpha, \beta, A\} = \{10^{-8}, 10^{-3}, 6 \times 10^{-4}\}$, so $P_2 = 10$, and $P_1 = 1.5$. In the figure, snapshots of the levels population (a,c,e,g) and the electric field (b,d,f,h) are shown for multiple τ , respectively. The inset also shows the total energy, $\sum_m \omega_m |\psi_m|^2$, which is seen to increase with time in a ladder-like manner. The moments of time at which $m \rightarrow m+1$ transitions occur agree with the theory, which predicts $\tau_m = mP_2$ [5]. The m th-level occupation numbers, $|\psi_m|^2$, also agree with the theoretical predictions, namely, $|\psi_m|^2 = \prod_{j=1}^{m-1} \mathcal{P}_{j \rightarrow j+1}$, where $\mathcal{P}_{j \rightarrow j+1}$ are given by Eq. (10). For example, at the adopted parameters, one has $|\psi_7(\tau = 55)|^2 = 0.83$, which deviates by only about 1% from the value obtained in the simulation.

A similar argument justifies our neglecting the terms of higher order in A in the derivation of Eq. (5). A direct calculation shows that those terms have the form $O(\omega_d E^1 A^2)$, so they cause additional “subharmonic” resonant transitions $m \rightarrow m+2$ with the effective driving parameter $\tilde{P}_1 \sim \beta A^2 / \sqrt{\alpha}$. The corresponding probability $\tilde{\mathcal{P}}_{m \rightarrow m+2}$ is given by an expression similar to Eq. (10), but P_1 should be replaced with \tilde{P}_1 [14]. For small enough A , one obtains $\tilde{\mathcal{P}}_{m \rightarrow m+2} \sim \beta^2 A^4 / \alpha \ll 1$. To have both $\mathcal{P}_{m \rightarrow m+1} \approx 1$ and $\tilde{\mathcal{P}}_{m \rightarrow m+2} \ll 1$, one must then require

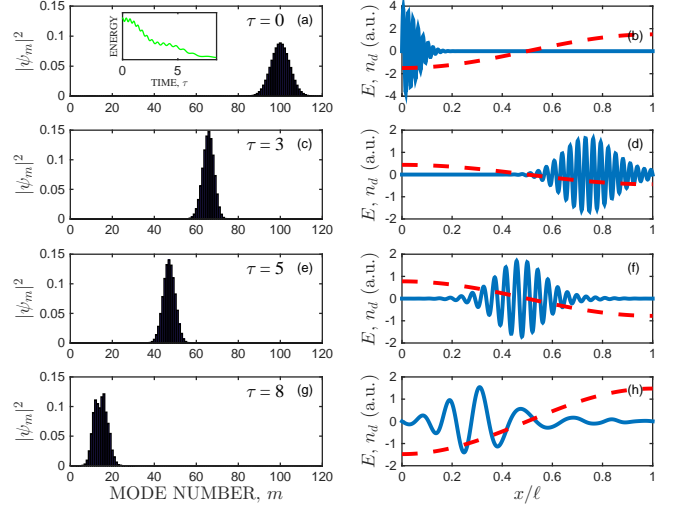


FIG. 3: (color online) Langmuir wave evolution in the down-chirped AR regime ($P_2 = 0.1$). The plotted quantities are the same as in Fig. 2, but the wave is initialized at larger m , and the chirp of the modulation frequency has the opposite sign.

the condition (which is satisfied in our simulations)

$$\alpha^{1/2} \lesssim A \ll \beta^{-1/2} \alpha^{1/4}. \quad (11)$$

Also note another, kinetic restriction of the LC mechanism. It stems from collisionless dissipation, which is not contained in our fluid equations. The local Landau damping rate for the m th mode in Maxwellian plasma is given by $\gamma_m \approx k_m^{-3} \sqrt{\pi/8} \exp(-k_m^{-2}/2 - 3/2)$ [24], where $k_m = m\sqrt{\beta/3}$, as before. Therefore, during the transition between neighboring levels, which occurs on the time scale $\Delta t = (\tau_{m+1} - \tau_m) / \sqrt{\alpha} = \beta/\alpha$, the energy decreases by the factor $\exp(2\Gamma_m)$, $\Gamma_m = \gamma_m \Delta t$. The value of Γ_m grows rapidly with m . Hence, just as the wave energy is shifted to the mode with $m = m_{\text{kin}}$, defined as that having $\Gamma_m \sim 1$, the energy is transferred to electrons almost momentarily, heating the tail distribution much like in Ref. [25]. At the parameters used in Fig. 1, $m_{\text{kin}} = 9$. This is larger than the maximum m attained in the simulation, so neglecting Landau damping is justified.

AR regime. — In contrast to the LC dynamics, in the limit $P_2 \ll 1$, many levels are coupled simultaneously. It can be shown then (e.g., using the Wigner phase space approach, as in Ref. [5]) that quantum LC continuously transforms into the classical AR by decreasing the effective Planck constant, P_2 . The electric field dynamics is then understood as AR acceleration of plasmons that satisfy $\omega_d = k_d v_g$, where

$$v_g = \frac{\omega_{m+1} - \omega_m}{k_{m+1} - k_m} \approx \frac{\partial \omega}{\partial k}. \quad (12)$$

Notably, this is the same “group-resonance” condition that was recently discussed in Ref. [26] (see Ref. [27] too). Also notably, the AR acceleration of plasmons

that we report here is akin to AR acceleration of resonant electrons in phase-mixed nonlinear waves such as Bernstein-Greene-Kruskal waves [8, 16, 28]. The AR dynamics of Langmuir waves is illustrated in Fig. 2, for which we adopted $\{\alpha, \beta, A\} = \{10^{-8}, 10^{-5}, 4.8 \times 10^{-3}\}$, so $P_2 = 0.1$. The simultaneous coupling of many levels is clearly seen in the left subplots, while the right subplots present the electric field of the autoresonant plasmon (solid blue lines) driven by the chirped density modulations, \tilde{n}_d (red dashed lines). In this case, Landau damping affects the total wave energy only by about 1% overall, so the kinetic restriction is not essential.

Interestingly, the dynamics is reversible, at least approximately. In fact, one can just as well capture a wave envelope at large k and then transport it down the spectrum, much like a trapped charged particle can be decelerated by a resonant field. We demonstrate the effect in Fig. 3, where the initial conditions are $E_0(x) \sim \exp(-(x - x_0)^2/2\sigma^2) \sin(k_{m_r}x)$, where m_r is the resonant wave number. In this example, we apply down-chirped driving phase $\varphi_d = \omega_{m_r, m_r+1}t - \alpha t^2/2 + \varphi_0$ with $\{\beta, x_0, \sigma, m_r, \alpha, A, \varphi_0\} = \{10^{-5}, 0, 0.05, 100, 10^{-8}, 0.004, \pi\}$, where φ_0 was chosen such that the plasmon is initially phase-locked with n_d .

Variational formulation. — Let us now recast our theory in a form that is not restricted to Langmuir waves but allows extending the above results to general nondissipative linear waves. Any such wave, described by some real field $\mathbf{E}(t, \mathbf{x})$ (electric field being an example), can be assigned a Lagrangian bilinear in \mathbf{E} , namely, of the form [3]

$$L = \int \mathbf{E} \cdot \hat{\mathcal{D}}(t, \mathbf{x}, i\partial_t, -i\nabla) \cdot \mathbf{E} d^3x. \quad (13)$$

The differential operator $\hat{\mathcal{D}}$ can be considered Hermitian without loss of generality. Suppose now that $\hat{\mathcal{D}} = \hat{\mathcal{D}}_0 + \hat{\mathcal{D}}_d$, where $\hat{\mathcal{D}}_0$ is some Hermitian operator that determines the Lagrangian in stationary homogeneous medium, and $\hat{\mathcal{D}}_d$ is a Hermitian operator that governs the mode interaction driven by some weak modulation. Let us represent the field as $\mathbf{E}(t, \mathbf{x}) = \text{Re} \sum_m \mathcal{E}_m(t) \mathbf{e}_m(\mathbf{x})$. Here \mathcal{E}_m are complex amplitudes and \mathbf{e}_m are orthonormal eigenmodes corresponding to the eigenfrequencies ω_m of $\hat{\mathcal{D}}_0$. Then, $L = \sum_m L_m + L_d$, where the two terms are, respectively, due to $\hat{\mathcal{D}}_0$ and $\hat{\mathcal{D}}_d$. Specifically [3],

$$L_m = \frac{i}{2} (\psi_m^* \dot{\psi}_m - \dot{\psi}_m^* \psi_m) - \omega_m |\psi_m|^2, \quad (14)$$

where the complex amplitudes ψ_m are defined such that $|\psi_m|^2$ is the action of the m th unperturbed mode; i.e., $\rho_m \psi_m = \mathcal{E}_m$, where $\rho_m^{-2} = d\bar{L}_m/d\omega_m$, $\bar{L}_m = (1/2) \int \mathbf{e}_m^* \cdot \hat{\mathcal{D}}_0 \cdot \mathbf{e}_m d^3x$, and ω_m are found by solving $\bar{L}_m(\omega_m) = 0$

[29]. Also, it is easy to see that

$$L_d \approx \sum_{m, m'} \psi_m^* h_{m, m'} \psi_{m'}, \quad (15)$$

$$h_{m, m'} = \frac{1}{2} \rho_m \rho_{m'} \int \mathbf{e}_m^* \cdot \hat{\mathcal{D}}_d \cdot \mathbf{e}_{m'} d^3x. \quad (16)$$

Then the equation for ψ_m has the form (7), and, since $h_{m, m'}$ is Hermitian, it manifestly conserves the wave total action, $\sum_m |\psi_m|^2$. For electromagnetic waves in particular, one has $\hat{\mathcal{D}}_d = \hat{\chi}_d/(8\pi)$, where $\hat{\chi}_d$ is the modulation-driven perturbation to the medium susceptibility. This can be seen, for instance, by comparing Eq. (13) with its geometrical-optics limit [29]. In the case of Langmuir waves, $\hat{\chi}_d = \hat{\mathcal{F}}$, and $\hat{\mathcal{D}}_0 = \hat{\epsilon}/(8\pi)$, where $\hat{\epsilon}$ is the dielectric permittivity operator; then one recovers Eq. (9). Finally, we note that it is straightforward to apply this approach to other classical waves [30]; hence, our further observations of LC and AR apply too.

Summary. — We report the first theoretical prediction of LC in a classical system. Specifically, we show that quasiperiodic chirped modulations of the background density can couple discrete eigenmodes of bounded plasma to produce controllable shift of the wave spectral energy distribution. Apart from academic interest, the new method of continuously controlling the wavelength of Langmuir oscillations might find practical applications, such as regulating coherent Raman scattering of laser radiation in plasma for generating short ultraintense pulses [31]. Our results indicate how similar techniques might be practiced with other plasma modes too, or in other settings such as waveguides or photonic crystals. Our work also bridges a number of effects that were previously considered unrelated. In particular, plasmon acceleration reported here can be seen as the resonant counterpart of the adiabatic ponderomotive effects on waves discussed recently in Ref. [26]. It is also akin to the resonant acceleration of charged particles trapped by chirped nonlinear plasma waves [16, 28]. Finally, our results further advance the general idea [1–3] that posing classical waves in quantumlike terms can be quite fruitful.

The work was supported by NNSA grant DE274-FG52-08NA28553, DOE contract DE-AC02-09CH11466, and DTRA grant HDTRA1-11-1-0037.

-
- [1] D. Dragoman and M. Dragoman, *Quantum-Classical Analogies* (Springer, Berlin, 2004).
 - [2] E. R. Tracy, A. J. Brizard, A. S. Richardson, and A. N. Kaufman, *Ray Tracing and Beyond: Phase Space Methods in Plasma Wave Theory* (Cambridge University Press, New York, 2014).
 - [3] I. Y. Dodin, Phys. Lett. A **378**, 1598 (2014).
 - [4] G. Marcus, L. Friedland, and A. Zigler, Phys. Rev. A **69**, 013407 (2004).

- [5] I. Barth, L. Friedland, O. Gat, and A. G. Shagalov, Phys. Rev. A **84**, 013837 (2011).
- [6] L. D. Landau, Phys. Z. Sowjetunion **2**, 46 (1932); C. Zener, Proc. R. Soc. London A **137**, 696 (1932).
- [7] M. Deutsch, B. Meerson, and J. E. Golub, Phys. Fluids B **3**, 1773 (1991); R. R. Lindberg, A. E. Charman, J. S. Wurtele, and L. Friedland, Phys. Rev. Lett. **93**, 055001(2004); G. B. Andresen et al. (ALPHA Collaboration), Phys. Rev. Lett. **106**, 025002 (2011); C. J. Baker, J. R. Danielson, N. C. Hurst, and C. M. Surko, Phys. Plasmas **22**, 022302 (2015).
- [8] L. Friedland, P. Khain, and A. G. Shagalov, Phys. Rev. Lett. **96**, 225001 (2006); I. Barth, L. Friedland, and A. G. Shagalov, Phys. Plasmas **15**, 082110 (2008).
- [9] O. Ben-David, M. Assaf, J. Fineberg, and B. Meerson, Phys. Rev. Lett. **96**, 154503 (2006).
- [10] O. Naaman, J. Aumentado, L. Friedland, J. S. Wurtele, and I. Siddiqi, Phys. Rev. Lett. **101**, 117005 (2008); K. W. Murch, R. Vijay, I. Barth, O. Naaman, J. Aumentado, L. Friedland, and I. Siddiqi, Nat. Phys. **7**, 105 (2011).
- [11] A. Barak, Y. Lamhot, L. Friedland, and M. Segev, Phys. Rev. Lett. **103**, 123901(2009); A. Barak and M. Segev, Phys. Rev. A **84**, 032123 (2011).
- [12] R. Malhotra, Nature (London) **365**, 819 (1993).
- [13] G. Marcus, A. Zigler, and L. Friedland, Europhys. Lett. **74**, 43 (2006); Y. Shalibo, Y. Rofo, I. Barth, L. Friedland, R. Bialczack, J. M. Martinis, and N. Katz, Phys. Rev. Lett. **108**, 037701 (2012).
- [14] I. Barth and L. Friedland, Phys. Rev. A **87**, 053420 (2013).
- [15] I. Barth and L. Friedland, Phys. Rev. Lett. **113**, 040403 (2014).
- [16] B. Breizman, Fusion Sci. Tech. **59**, 549 (2011); P. F. Schmit, I. Y. Dodin, J. Rocks, and N. J. Fisch, Phys. Rev. Lett. **110**, 055001 (2013).
- [17] P. F. Barker and M. N. Shneider, Phys. Rev. A **66**, 065402 (2002); M. N. Shneider, S. F. Gimelshein, and P. F. Barker, J. Appl. Phys. **99**, 063102 (2006); M. N. Shneider, P. F. Barker, and S. F. Gimelshein, Appl. Phys. A **89**, 337 (2007); F. Peano, J. Vieira, R. Mulas, G. Coppa, R. Bingham, and L. O. Silva, Plasma Phys. Control. Fusion **51**, 024006 (2009).
- [18] The omitted term is of the form $O(\omega_d E)$. Under the assumed conditions, keeping it would modify the mode-coupling coefficients below by factor $1 + O(\omega_d)$. This correction is negligible within the adopted accuracy, because we further assume $\omega_d = O(\beta)$ and entirely neglect thermal corrections in Eq. (9).
- [19] I. Y. Dodin, V. I. Geyko, and N. J. Fisch, Phys. Plasmas **16**, 112101 (2009).
- [20] Our system differs from the anharmonic quantum oscillator [5] just by the values of ω_m and ρ_m .
- [21] R. L. Dewar, J. Plasma Phys. **7**, 267 (1972).
- [22] E. Nakar and L. Friedland, Phys. Rev. E **60**, 5479 (1999).
- [23] In an infinitely deep square potential, a quantum particle (or, equivalently, classical wave) can be captured into AR even at the lowest energy state, since the corresponding eigenfrequency is nonzero. This is in contrast with the classical model of AR in such systems, within which automatic capturing into AR at the very bottom of an infinitely deep square potential is impossible [22].
- [24] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Pergamon Press, New York, 1981), Sec. 32.
- [25] P. F. Schmit, I. Y. Dodin, and N. J. Fisch, Phys. Rev. Lett. **105**, 175003 (2010).
- [26] I. Y. Dodin and N. J. Fisch, Phys. Rev. Lett. **112**, 205002 (2014).
- [27] V. N. Tsytovich, *Nonlinear Effects in Plasma* (Plenum Press, New York, 1970).
- [28] V. L. Krasovsky, Phys. Lett. A **163**, 199 (1992); V. L. Krasovskii, Zh. Eksp. Teor. Fiz. **107**, 741 (1995) [JETP **80**, 420 (1995)]; W. Bertsche, J. Fajans, and L. Friedland, Phys. Rev. Lett. **91**, 265003 (2003); I. Y. Dodin and N. J. Fisch, Phys. Plasmas **19**, 012104 (2012).
- [29] I. Y. Dodin and N. J. Fisch, Phys. Rev. A **86**, 053834 (2012).
- [30] Related equations for coupling of specific waves in modulated media appear, for instance, in J. N. Winn, S. Fan, J. D. Joannopoulos, and E. P. Ippen, Phys. Rev. B **59**, 1551 (1999); I. Y. Dodin and N. J. Fisch, Phys. Plasmas **9**, 760 (2002); Phys. Rev. Lett. **88**, 165001 (2002).
- [31] N. J. Fisch and V. M. Malkin, Phys. Plasmas **10**, 2056 (2003).